SID: 1820252123 謝仁和 100

1. Solve the initial-value problem $(1+\cos x)y' = (1+e^{-y})\sin x, \ y(0) = 0$.

40

$(1+\cos x)y' = (1+e^{-y})\sin x$

$(1+\cos x)y' - e^{-y}\sin x = \sin x$

$y' - \frac{e^{-y}\sin x}{1+\cos x} = \frac{\sin x}{1+\cos x}$

$\frac{dy}{dx} = \frac{(1+e^{-y})\sin x}{1+\cos x}$

$\frac{dy}{1+e^{-y}} = \frac{(\sin x)}{1+\cos x}dx$

$u = e^y, \ du = e^y dy, \ dy = \frac{du}{u}$ $\int \frac{1}{1+e^{-y}}dy = \int \frac{\sin x}{1+\cos x}dx, \ u = 1+\cos x, \ du = -\sin x\,dx$

$= \int \frac{\frac{du}{u}}{1+\frac{1}{u}} = -\int \frac{1}{u}du$

$= \int \frac{du}{u+1} = -\ln|u| + c$

$= \ln|u+1| = -\ln|u| + c, \ \ln|e^y+1| = -\ln|1+\cos x| + c, \ e^y + 1 = \frac{c}{1+\cos x}$

$\to y(0) = 0, \ e^0 + 1 = \frac{c}{1+\cos 0}$

$1 + 1 = \frac{c}{1+1}$

$2 = \frac{c}{2}, \ C = 4$

$e^y + 1 = \frac{4}{1+\cos x}$

$e^y = \frac{4}{1+\cos x} - 1 = \frac{3-\cos x}{1+\cos x}$

$y(x) = \ln\left(\frac{3-\cos x}{1+\cos x}\right)$ ✓

2. A spring with a mass of 2 kg has damping constant 16, and a force of 12.8 N keeps the spring stretched 0.2 m beyond its natural length. Find the position of the mass at time t if it starts at the equilibrium position with a velocity of 2.4 m/s.

60

$m = 2kg, \ F = 12{,}8N, \ x = 0{,}2, \ F = kx, \ 12{,}8 = k\cdot 0{,}2, \ k = 64N/m, \ c = 16$

$mx'' + cx' + kx = 0$

$2x'' + 16x' + 64x = 0$

$x'' + 8x' + 32x = 0$

$r^2 + 8r + 32 = 0, \ r = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-8 \pm \sqrt{64-128}}{2} = -4 \pm 4i, \ \alpha = -4, \ \beta = 4$

$x(t) = e^{-4t}(c_1\cos 4t + c_2\sin 4t)$

$x(0) = 0,$

$0 = e^0(c_1\cos 0 + c_2\sin 0), \ 0 = 1\cdot(c_1\cdot 1 + 0), \ c_1 = 0$

$x(t) = c_2\cdot e^{-4t}\cdot\sin 4t, \ x'(t) = c_2\cdot e^{-4t}(4\cos 4t - 4\sin 4t)$

$v(0) = x'(0) = 2{,}4$

$2{,}4 = c_2\cdot e^0(4\cos 0 - 4\sin 0), \ 2{,}4 = c_2\cdot 1(4\cdot 1 - 0), \ 2{,}4 = 4c_2, \ c_2 = 0{,}6$

$x(t) = c_2 e^{-4t}\sin 4t = 0{,}6\cdot e^{-4t}\cdot\sin 4t$ ✓

100

SID: 1820252123 **Name:** 謝仁和

1. Find the points on the hyperboloid $x^2 + 4y^2 - z^2 = 4$, where the tangent plane is parallel to the plane $2x + 2y + z = 5$. $V = \langle 2, 2, 1 \rangle$

$f_x = 2x$, $\nabla F = kV$

$f_y = 8y$ $\quad \langle 2x_0, 8y_0, -2z_0 \rangle = k\langle 2, 2, 1 \rangle$

$f_z = -2z$

$\nabla F = \langle f_x, f_y, f_z \rangle$ $\quad 2x_0 = 2k,\ x_0 = k =$

$= \langle 2x, 8y, -2z \rangle$ $\quad 8y_0 = 2k,\ y_0 = \frac{1}{4}k$

$-2z_0 = k,\ z_0 = -\frac{1}{2}k$

$= \langle k, \frac{1}{4}k, -\frac{1}{2}k \rangle$

$4 = k^2 + 4\cdot\left(\frac{1}{4}k\right)^2 - \left(-\frac{1}{2}k\right)^2$

$4 = k^2 + \cancel{\frac{1}{4}k^2} - \cancel{\frac{1}{4}k^2}$

$k^2 = 4,\ k = \sqrt{4} = \pm 2$

$\left(2, \frac{1}{2}, -1\right)\ (k = 2)$

or

$\left(-2, -\frac{1}{2}, 1\right)(k = -2)$

25

2. If $u = x^2y^3 + z^4$, where $x = p + 3p^2, y = pe^p$, and $z = p\sin p$, use the Chain Rule to find $\frac{du}{dp}$.

$\frac{du}{dp} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dp} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dp} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dp} = 2xy^3(1+6p) + 3x^2y^2(e^p + pe^p) + 4z^3(\sin p + p\cos p)$

$\frac{\partial u}{\partial x} = 2xy^3,\ \frac{\partial u}{\partial y} = 3x^2y^2,\ \frac{\partial u}{\partial z} = 4z^3$

$\frac{dx}{dp} = 1 + 6p,\ \frac{dy}{dp} = e^p + pe^p,\ \frac{dz}{dp} = \sin p + p\cos p$

25

3. Find the directional derivative of $f(x, y, z) = x^2y + x\sqrt{1+z}$ at the given point $(1, 2, 3)$, in the direction $\mathbf{v} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$. , $V = \langle 2, 1, -2 \rangle$

$f_x = 2xy + \sqrt{1+z},\ f_x(1,2,3) = 2\cdot1\cdot2 + \sqrt{1+3} = 4 + 2 = 6$

$f_y = x^2,\ f_y(1,2,3) = 1^2 = 1$

$f_z = \frac{x}{2\sqrt{1+z}},\ f_z(1,2,3) = \frac{1}{2\sqrt{1+3}} = \frac{1}{2\sqrt{4}} \pm \frac{1}{2\cdot2} = \frac{1}{4}$

$\langle 6, 1, \frac{1}{4} \rangle = \nabla F(1,2,3)$

$\nabla f(1,2,3) = \langle f_x, f_y, f_z \rangle = \langle 6, 1, \frac{1}{4} \rangle$

$\|v\| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{4+1+4} = \sqrt{9} = 3$

$u = \frac{1}{3}\langle 2, 1, -2 \rangle = \langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \rangle$

$D_u = \nabla F \cdot u$

$D_u = \frac{2}{3}\cdot 6 + \frac{1}{3}\cdot 1 + -\frac{\cancel{2}^{1}}{3}\cdot\frac{1}{\cancel{4}_{2}}$

$= \frac{12}{3} + \frac{1}{3} - \frac{1}{6}$

$= 4 + \frac{2}{6} - \frac{1}{6}$

$= 4 + \frac{1}{6} = \frac{25}{6}$

25

4. Find the gradient of the function $f(x, y, z) = x^2e^{yz^2}$.

$f_x = 2xe^{yz^2}$

$f_y = x^2z^2e^{yz^2}$

$f_z = x^2\cdot 2z\cdot e^{yz^2}\cdot y$

$\nabla F = \langle f_x, f_y, f_z \rangle = \langle 2xe^{yz^2}, x^2z^2e^{yz^2}, 2x^2yze^{yz^2} \rangle$

25

謝仁和

100

SID: 1820152123 **Notice: The numbers have minor differences from the original questions.**

1.Find the point in which the line with parametric equations $x = 1 + 3t,\ y = 2 - t,\ z = 4t$ intersects the plane $-x + 2y + z = 2$.

$-(1+3t) + 2(2-t) + 4t = 2$

$~~t~~ -1-3t+4-2t+4t = 2$

$-t = -1,\ t = 1$

$x = 1+3\cdot 1 = 1+3 = 4$

$y = 2-1 = 1$

$z = 4\cdot 1 = 4$

$(4,1,4)$ //

25

2. (a) Find an equation of the sphere that passes through the point $(6,\ 3,\ -2)$ and has center $(-1, -1,\ 2)$.

(b) Find the curve in which this sphere intersects the yz −plane. yz plane, $x=0$

$A(6,3,-2),\ B(-1,-1,2).\ \overrightarrow{AB} = B-A = (-1-6, -1-3, 2+2) = (-7,-4,4)$

$(x+1)^2 + (y+1)^2 + (z-2)^2 = r^2$

$(6+1)^2 + (3+1)^2 + (-2-2)^2$

$= 49 + 16 + 16 = 32+49 = 81$

a. $(x+1)^2 + (y+1)^2 + (z-2)^2 = 81$ //

b. $x=0,\ 1+(y+1)^2 + (z-2)^2 = 81$

$(y+1)^2 + (z-2)^2 = 80$ //

25

3. (a) Find a vector perpendicular to the plane through the points $A(2,0,-1),\ B(1,0,0)$, and $C(1,4,3)$.

(b) Find the area of triangle ABC.

a. $\overrightarrow{AB} = B-A = (1-2,\ 0-0,\ 0+1) = (-1,0,1)$

$\overrightarrow{AC} = C-A = (1-2,\ 4-0,\ 3+1) = (-1,4,4)$

$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ -1 & 4 & 4 \end{vmatrix}\begin{matrix} i & j \\ -1 & 0 \\ -1 & 4 \end{matrix} = 0 - j - 4k + 0 - 4i + 4j$

$= -4i + 3j - 4k = (-4,3,-4) = n$

$A(2,0,-1)\ a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

$-4(x-2) + 3(y-0) + -4(z+1) = 0$

$-4x+8+3y-4z-4 = 0,\ -4x+3y-4z = -4$ //

b. At the back

~~b. Area = $\frac{1}{2}$ ... $= \frac{1}{2}(0-1-4+0-4+4) = \frac{1}{2}\cdot\sqrt{5} = 2.5 = \frac{5}{2}$ units~~

25

4. Let $\mathbf{r}(t) = \langle \frac{e^t-1}{t},\ \sqrt{2-t},\ \ln(t+1) \rangle$

(a) Find the domain of $\mathbf{r}$.

(b) Find $\lim_{t\to 0}\mathbf{r}(t)$.

(c) Find $\mathbf{r}'(t)$.

a → $-1 < t \leq 2,\ t \neq 0$ //

$t \leq 2$?

~~$-1 \leq t \leq 2,\ t \neq 0$~~

a. $x = \frac{e^t-1}{t},\ t \neq 0,\ y = \sqrt{2-t},\ t,\ z = \ln(t+1),$ $t > -1$

b. $\lim_{t\to 0} r(t) = \left(\lim_{t\to 0}\frac{e^t-1}{t},\ \lim_{t\to 0}\sqrt{2-t},\ \lim_{t\to 0}\ln(t+1)\right) = (1, \sqrt{2}, 0)$ //

25

c. $r'(t) = \left\langle \frac{d}{dt}\left(\frac{e^t-1}{t}\right),\ \frac{d}{dt}\left(\sqrt{2-t}\right),\ \frac{d}{dt}\ln(t+1)\right\rangle$

$\frac{d}{dt}\left(\frac{e^t-1}{t}\right),\ u = e^t-1,\ u' = e^t,\ v = t,\ v' = 1\ \ \frac{d}{dt} = \frac{u'v-uv'}{v^2} = \frac{te^t-e^t+1}{t^2}$